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Two-phase displacement in Hele Shaw cells: theory

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A theory describing two-phase displacement in the gap between closely spaced planes is developed. The main assumptions of the theory are that the displaced fluid wets the walls, and that the capillary number Ca and the ratio of gap width to transverse characteristic length ϵ are both small. Relatively mild restrictions apply to the ratio M of viscosities of displacing to displaced fluids; in particular the theory holds for $M = o(Ca^{-\frac{1}{3}})$. We formulate the theory as a double asymptotic expansion in the small parameters ϵ and $Ca^{\frac{1}{3}}$. The expansion in ϵ is uniform while that in $Ca^{\frac{1}{3}}$ is not, necessitating the use of matched asymptotic expansions. The previous work of Bretherton (1961) is clarified and extended, and both the form and the constants in the effective boundary condition of Chouke, van Meurs & van der Poel (1959) and of Saffman & Taylor (1958) are determined.

1. Introduction

The Hele Shaw cell involving flow in thin gaps, since its description and exposition by Hele Shaw (1898), has proven to be a useful analogue for visualization and description of potential fields (Van Dyke 1982; Moore 1949). It has also been used to study two-phase displacements, and has provided useful insight into viscously driven instability phenomena (Chouke, van Meurs & van der Poel 1959; Saffman & Taylor 1958). In addition, certain model free-boundary problems can be posed for Hele Shaw flows and solved using free-streamline theory and the powerful techniques of potential theory (Saffman & Taylor 1958; McLean & Saffman 1981). However, in spite of a considerable amount of work on the solution of such two-phase flow problems, a rigorous derivation of the equations and boundary conditions has not been completed to date. As is well known, Hele Shaw theory for *single-phase flow* results in the following equations relating the *depth-averaged* pressure and two-dimensional velocity fields:

$$\nabla \cdot \bar{\boldsymbol{u}} = 0, \tag{1.1a}$$

$$\nabla p = -A\mu \bar{\boldsymbol{u}} - \rho \boldsymbol{g}, \tag{1.1b}$$

with $A = 3/b^2$, where b is the gap *half*-width. Saffman & Taylor (1958) have discussed the modification of Hele Shaw theory for the special case in which there is a wetting film of constant thickness on the solid surface when there are two phases present. Equations of the form of (1.1) still hold for the flow in two-phase regions, with the modifications that the constant A and the effective density ρ depend upon the viscosity ratio and the thickness ratio of the phases.

Of course, in the case of simultaneous flow of two phases, it is necessary to write (1.1) in each phase, and boundary conditions must be given which hold for the *depth-averaged fields*. It is common to write these as

$$[\![\boldsymbol{n}\cdot\bar{\boldsymbol{u}}]\!] = 0, \tag{1.2}$$

$$[\![p]\!] = \frac{\gamma}{b}\cos\theta + \frac{\gamma}{R}. \tag{1.3}$$

Here $[\![\ ]\!]$ denotes a jump, γ is the surface tension, θ the apparent contact angle, and R the principal radius of curvature of the projection, onto the plane, of the tip of the meniscus separating the two phases (see figure 1).

Equation (1.2) is obvious from the kinematics of the flow, but it is not clear that (1.3) is correct, except at equilibrium under conditions of no flow. Equation (1.3) appears to have been first suggested by Chouke *et al.* (1959), and was adopted by Saffman & Taylor (1958), but not without reservation. These latter authors qualify their use of it (which was *not* extensive), by saying 'The effect of surface tension on the stability of the interface may depend upon a variety of physical conditions. The *simplest assumption* is to take the pressure drop through the interface as $\gamma(1/b+1/R)$ where R is the radius of curvature of the projection on the planes bounding the cell of the tip of the meniscus.' (Emphasis added.)

Indeed, the recent work of McLean & Saffman (1981) has cast serious doubt on the validity of this boundary condition, and Saffman (1982) has carefully and lucidly discussed the issues involved. Thus an explicit derivation of the correct boundary condition is one of the primary concerns of this paper.

This derivation is complicated in detail, as it involves the solution of the underlying free boundary problem. In order to proceed from a well-defined problem, it is necessary to make assumptions regarding the wetting conditions. We assume below that the displaced fluid wets the wall, so that problems involved with modelling the moving contact line are circumvented. Furthermore, in order to make analytical progress, it is necessary to hinge a perturbation scheme about a limit for which the solution of the free boundary problem is tractable. As we shall see, the limit of small capillary number or small dimensionless displacement velocity is the appropriate one. We are thus led to reconsider a problem originally discussed by Bretherton (1961) involving the propagation of an inviscid bubble into a viscous fluid at small capillary number. Interestingly, this problem is also related to other free-surface problems at low capillary number, and certain of Bretherton's results were preceded by the ideas of Landau & Levich (1942), who discussed coating flows. Although the essential ideas implicit in these early works are those of matched solutions, neither paper is written with the modern techniques of matched asymptotic expansions at hand. In this regard, the papers of Ruschak & Scriven (1977) on flow from a slot and Wilson (1982) on coating flows for low capillary number provide useful background. We have found it necessary, in developing a rigorous theory of three-dimensional two-phase flow in Hele Shaw cells, to construct a formal expansion procedure in order to derive the correct form of (1.3). In so doing, we clarify and strengthen certain results of Bretherton (1961). It is our hope that our exposition will also help readers to understand the earlier works as well.

In §2 we pose the free-boundary problem under consideration. Section 3 discusses the relevant scalings and the need for the use of a second set of lubrication layer scalings because of the non-uniformity of the zero-capillary-number limit. Section 4 treats one-dimensional displacement and, in addition to the clarification of earlier works alluded to above, provides an extension of the results of Bretherton (1961). Finally, in §5 we treat the full three-dimensional moving-boundary problem in the Hele Shaw approximation to derive the asymptotic form of the jump condition (1.3) valid for small capillary number and thin gaps.

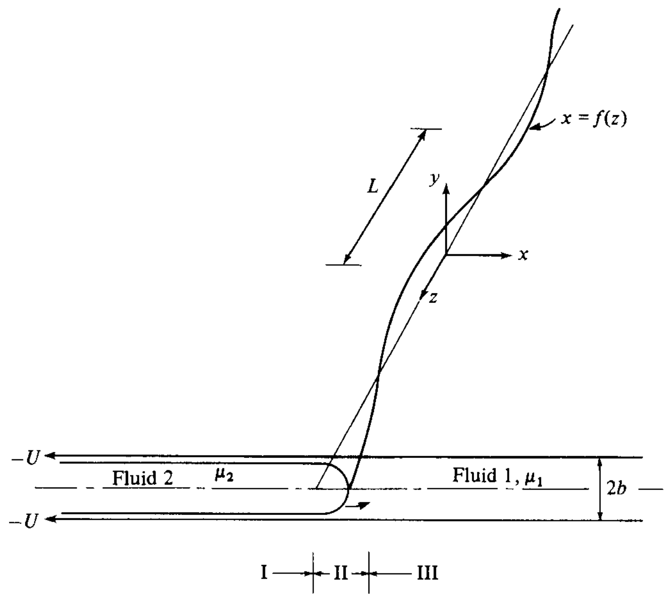

FIGURE 1. Two-phase flow in a Hele Shaw cell. The interface is located symmetrically at $y = \pm h(x, z)$.

2. Basic equations

We shall be concerned with slow displacement of one fluid by another in a Hele Shaw cell. We consider the displacement to take place with constant velocity U, and the motion of both fluids to be in the Stokes regime. Thus the steady equations of change and boundary conditions are as follows:

$$\left.\begin{aligned}\nabla\cdot\boldsymbol{u}_1 &= 0,\\ \nabla P_1 &= \mu_1\nabla^2\boldsymbol{u}_1,\end{aligned}\right\} \tag{2.1a}$$

$$\left.\begin{aligned}\nabla\cdot\boldsymbol{u}_2 &= 0,\\ \nabla P_2 &= \mu_2\nabla^2\boldsymbol{u}_2,\end{aligned}\right\} \tag{2.1b}$$

$$u_1 = -U, \quad v_1 = w_1 = 0 \quad \text{at} \quad y = \pm b, \tag{2.2}$$

$$\left.\begin{aligned}\boldsymbol{u}_1 &= \boldsymbol{u}_2, & (2.3a)\\ \boldsymbol{n}\cdot\boldsymbol{u} &= 0, & (2.3b)\\ \boldsymbol{t}\cdot(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\boldsymbol{n} &= 0, & (2.3c)\\ \boldsymbol{n}\cdot(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\boldsymbol{n} &= \gamma(\nabla\cdot\boldsymbol{n}) & (2.3d)\end{aligned}\right\} \quad \text{at} \quad y = \pm h(x,z),$$

$$v = 0, \quad u_y = 0 \quad \text{at} \quad y = 0. \tag{2.4}$$

Here the subscripts refer to quantities within each fluid, (u, v, w) are the components of $\boldsymbol{u}$ in the (x, y, z)-directions, and figure 1 shows a schematic of this three-dimensional moving-boundary problem. Note that the problem formulation is written in a reference frame moving with the displacement velocity U. b and $h(x, z)$ represent the half-thickness of the Hele Shaw cell and the shape of the interface respectively. $\boldsymbol{t}$ and

$\boldsymbol{n}$ represent the unit tangent and unit normal vectors to the interface and γ the interfacial tension. $f(z)$ represents the projection of the tip of the interface onto the (x, z)-plane. The regions labelled I, II, III are explained below. Our problem is to solve for the relationship between the velocity and pressure fields and $h(x, z)$.

We assume that the displaced fluid totally wets the wall, leaving a film on the wall as displacement proceeds. Furthermore, we have neglected gravity (a suitable Bond number is small) so that the solution is symmetric about the midplane $y = 0$.

When the equations and boundary conditions (2.1)–(2.4) are non-dimensionalized, they are governed by the following three independent parameters:

$$M = \frac{\mu_2}{\mu_1}, \quad \epsilon = \frac{b}{L}, \quad Ca = \frac{\mu_1 U}{\gamma}.$$

M is the viscosity ratio of the two immiscible fluids, ϵ is the ratio of the two lengthscales b and L, which are the half-thickness of the Hele Shaw cell and the characteristic length of the lateral variation respectively, and Ca is the capillary number, which represents the relative magnitude of the viscous force to the interfacial tension. Because the thickness of the Hele Shaw cell is very small and the flow is very slow, both ϵ and Ca are very small quantities. Even though the full problem is very difficult to solve, it can be treated by developing a double-expansion method in ϵ and Ca. As we shall see, however, the limit $Ca \rightarrow 0$ is not a uniform one, and we shall require use of matched asymptotic expansions in order to treat it.

3. Scalings and regions

The domain is divided into three regions: the constant-film-thickness region (region I), the front-meniscus region (region II) and the clear-original-fluid region (region III). In region III the flow is parabolic in y, and classical Hele Shaw theory applies. In region I the problem can be solved by the method of regular perturbation expansion in ϵ and Ca if the film thickness is given. This analysis is given in the appendix of Saffman & Taylor (1958), and will not be repeated here, as it leads to a straightforward modification of Hele Shaw theory. A detailed analysis of region II is required to complete the problem.

In region II, because the capillary number is small, we begin by setting it to zero, with the result that the interface is almost hydrostatic, and its shape is thus nearly circular. Pressure and interfacial tension are important in that region. This cannot be a uniform solution, however, since it cannot be smoothly matched to the constant-thickness solution of region I. Thus there exists a *transition* region in which the shape of the interface is deformed by viscous traction and thus viscous forces also become important. Therefore the solution by regular perturbation expansion in Ca will not be uniformly valid throughout region II, and the method of matched asymptotic expansions should be used between two subregions; the capillary-statics region where the shape of the interface is almost circular and the transition region where the lubrication approximation can be applied.

It is necessary to examine region II more carefully. In the capillary-statics region the equation of change and boundary conditions can be non-dimensionalized by the following scales:

$$(x, y, z) \sim (b, b, L), \quad \boldsymbol{u} \sim U, \quad P \sim \frac{\gamma}{b}, \quad h \sim b.$$

Using these scales, the equation of change and boundary conditions become (see Atherton & Homsy, 1976):

$$\left.\begin{aligned} u_{1x}+v_{1y}+\epsilon w_{1z} &= 0,\\ P_{1x} &= Ca(u_{1xx}+u_{1yy}+\epsilon^2 u_{1zz}),\\ P_{1y} &= Ca(v_{1xx}+v_{1yy}+\epsilon^2 v^{1zz}),\\ \epsilon P_{1z} &= Ca(w_{1xx}+w_{1yy}+\epsilon^2 w_{1zz}),\end{aligned}\right\} \qquad (3.1a)$$

$$\left.\begin{aligned} u_{2x}+u_{2y}+\epsilon w_{2z} &= 0,\\ P_{2x} &= M\,Ca(u_{2xx}+u_{2yy}+\epsilon^2 u_{2zz}),\\ P_{2y} &= M\,Ca(v_{2xx}+v_{2yy}+\epsilon^2 v_{2zz}),\\ \epsilon P_{2z} &= M\,Ca(w_{2xx}+w_{2yy}+\epsilon^2 w_{2zz}),\end{aligned}\right\} \qquad (3.1b)$$

$$u_1 = -1, \quad v_1 = 0, \quad w_1 = 0 \quad \text{at} \quad y = -1, \qquad (3.2)$$

$$u_1 = u_2, \quad v_1 = v_2, \quad w_1 = w_2 \quad \text{at} \quad y = -h(x,z), \qquad (3.3a)$$

$$u_1 h_x + \epsilon w h_z = -v \quad \text{at} \quad y = -h(x,z), \qquad (3.3b)$$

$$\left.\begin{aligned} &[\![\mu\{(\epsilon h_z u_y - h_x w_y) - 2\epsilon h_x h_z(\epsilon w_z - u_x) - (\epsilon u_z + w_x)\,(h_x^2 - \epsilon^2 h_z^2)\\ &\qquad + \epsilon(h_z v_x - h_x v_z)\}]\!] = 0,\\ &[\![\mu[(h_x^2+\epsilon^2 h_z^2)\{-h_x(u_y+v_x) - \epsilon h_z(\epsilon v_z + w_y)\} + 2\{h_x^2(u_x - v_y)\\ &\quad + \epsilon^2 h_z^2(\epsilon w_z - v_y) + \epsilon h_x h_z(\epsilon u_z + w_x)\}\\ &\qquad + h_x(u_y+v_x) + \epsilon h_z(\epsilon v_z + w_y)]]\!] = 0 \end{aligned}\right\} \text{at} \quad y = -h(x,z), \qquad (3.3c)$$

$$\begin{aligned} &P_2 - P_1 - \frac{2Ca}{(1+h_x^2+\epsilon^2 h_z^2)}[\![\mu\{-h_x(u_y+v_x) - \epsilon h_z(\epsilon v_z + w_y)\\ &- \epsilon h_x h_z(\epsilon u_z + w_x) - v_y - h_x^2 u_x - \epsilon^3 h_z^2 w_z\}]\!]\\ &\quad = \frac{-1}{(1+h_x^2+\epsilon^2 h_z^2)^{\frac{3}{2}}}\{h_{xx}(1+\epsilon^2 h_z^2) + \epsilon^2 h_{zz}(1+h_x^2)\\ &\quad - 2\epsilon^2 h_x h_z h_{xz}\} \quad \text{at} \quad y = -h(x,z), \end{aligned} \qquad (3.3d)$$

$$u_y = 0, \quad v = 0 \quad at \quad y = 0. \qquad (3.4)$$

Here the subscripts x, y, z represent partial differentiation and $[\![\boldsymbol{u}]\!]$ represents the jump $\boldsymbol{u}_1 - \boldsymbol{u}_2$. The dimension*less* viscosities are so defined that $\mu_1 = 1$ and $\mu_2 = M$.

In the transition region, the variables should be rescaled, because in this region the lengthscales change such that the viscous force becomes as important as pressure or interfacial tension. The new scales can be determined by examining the equation of motion for fluid 1 in the lubrication approximation and retaining all the three equally important forces: pressure, interfacial tension and viscous force. The new

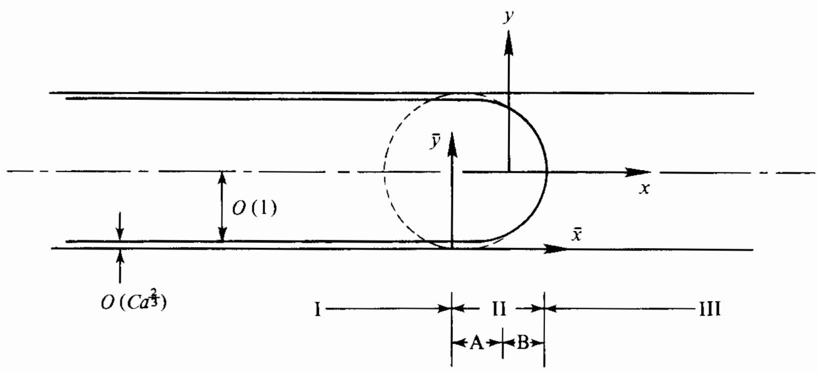

FIGURE 2. Schematic showing the scales and coordinate systems for the different regions. (A: transition region, B: capillary-statics region).

variables, rescaled for the transition region and denoted by an overbar, are

$$(\bar{x}_1, \bar{y}_1, \bar{z}_1) = \left(\frac{x+l}{Ca^{\frac{1}{3}}}, \frac{y+1}{Ca^{\frac{2}{3}}}, z\right),$$

$$(\bar{u}_1, \bar{v}_1, \bar{w}_1) = \left(u, \frac{v}{Ca^{\frac{1}{3}}}, w\right),$$

$$(\bar{x}_2, \bar{y}_2, \bar{z}_2) = \left(\frac{x+l}{Ca^{\frac{1}{3}}}, y+1, z\right),$$

$$(\bar{u}, \bar{v}_2, \bar{w}_2) = (u, Ca^{\frac{1}{3}}v, w),$$

$$\bar{P} = P, \quad \bar{h} = (1-h)\,Ca^{-\frac{2}{3}}.$$

For fluid 2 there is no reason for rescaling the y-coordinate, so it is not stretched. $x = -l$ is the location of origin of the coordinate system for the transition layer, and it will be determined by matching conditions. The locations of the origins of each region and their relative lengths are shown in figure 2.

Using the rescaled variables, the equation of change and boundary conditions in the transition layer become

$$\left.\begin{aligned} \bar{u}_{1\bar{x}} + \bar{v}_{1\bar{y}} + \epsilon Ca^{\frac{1}{3}}\bar{w}_{1\bar{z}} &= 0, \\ \bar{P}_{1\bar{x}} &= Ca^{\frac{2}{3}}\bar{u}_{1\bar{x}\bar{x}} + \bar{u}_{1\bar{y}\bar{y}} + \epsilon^2 Ca^{\frac{4}{3}}\bar{u}_{1\bar{z}\bar{z}}, \\ \bar{P}_{1\bar{y}} &= Ca^{\frac{4}{3}}\bar{v}_{1\bar{x}\bar{x}} + Ca^{\frac{2}{3}}\bar{v}_{1\bar{y}\bar{y}} + \epsilon^2 Ca^{2}\bar{v}_{1\bar{z}\bar{z}}, \\ \epsilon Ca^{\frac{1}{3}}\bar{P}_{1\bar{z}} &= Ca^{\frac{2}{3}}\bar{w}_{1\bar{x}\bar{x}} + \bar{w}_{1\bar{y}\bar{y}} + \epsilon^2 Ca^{\frac{4}{3}}\bar{w}_{1\bar{z}\bar{z}}, \end{aligned}\right\} \tag{3.5a}$$

$$\left.\begin{aligned} \bar{u}_{2\bar{x}} + \bar{v}_{2y} + \epsilon Ca^{\frac{1}{3}}\bar{w}_{2\bar{z}} &= 0, \\ \bar{P}_{2\bar{x}} &= M(Ca^{\frac{2}{3}}\bar{u}_{2\bar{x}\bar{x}} + Ca^{\frac{4}{3}}\bar{u}_{2yy} + \epsilon^2 Ca^{\frac{4}{3}}\bar{u}_{2\bar{z}\bar{z}}), \\ \bar{P}_{2y} &= M(\bar{v}_{2\bar{x}\bar{x}} + Ca^{\frac{2}{3}}\bar{v}_{2yy} + \epsilon^2 Ca^{\frac{2}{3}}\bar{v}_{2\bar{z}\bar{z}}), \\ \epsilon Ca^{\frac{1}{3}}\bar{P}_{2\bar{z}} &= M(Ca^{\frac{2}{3}}\bar{w}_{2\bar{x}\bar{x}} + Ca^{\frac{4}{3}}\bar{w}_{2yy} + \epsilon^2 Ca^{\frac{4}{3}}\bar{w}_{2\bar{z}\bar{z}}), \end{aligned}\right\} \tag{3.5b}$$

$$\bar{u}_1 = -1, \quad \bar{v}_1 = 0, \quad \bar{w}_1 = 0 \quad \text{at} \quad \bar{y} = 0, \tag{3.6}$$

$$\bar{u}_1 = \bar{u}_2, \quad Ca^{\frac{2}{3}}\bar{v}_1 = \bar{v}_2, \quad \bar{w}_1 = \bar{w}_2 \quad \text{at} \quad \bar{y} = \bar{h}(\bar{x}, \bar{z}), \tag{3.7a}$$

$$\bar{u}_1 \bar{h}_{\bar{x}} + \epsilon Ca^{\frac{1}{3}}\bar{w}_1 \bar{h}_{\bar{z}} = \bar{v}_1 \quad \text{at} \quad \bar{y} = \bar{h}(\bar{x}, \bar{z}), \tag{3.7b}$$

$$[(\epsilon Ca^{\frac{1}{3}}\bar{h}_{\bar{z}}\bar{u}_{1\bar{y}}-\bar{h}_{\bar{x}}\bar{w}_{1\bar{y}})+2\epsilon Ca\bar{h}_{\bar{x}}\bar{h}_{\bar{z}}(\epsilon Ca^{\frac{1}{3}}\bar{w}_{1\bar{z}}-\bar{u}_{1\bar{x}})$$
$$+Ca^{\frac{2}{3}}(\epsilon Ca^{\frac{1}{3}}\bar{u}_{1\bar{z}}+\bar{w}_{1\bar{x}})(\bar{h}_{\bar{x}}^2-\epsilon^2 Ca^{\frac{2}{3}}\bar{h}_{\bar{z}}^2)+\epsilon Ca(\bar{h}_{\bar{z}}\bar{v}_{1\bar{x}}-\bar{h}_{\bar{x}}\bar{v}_{1\bar{z}})]$$
$$=M\,Ca^{\frac{2}{3}}[(\epsilon Ca^{\frac{1}{3}}\bar{h}_{\bar{z}}\bar{u}_{2y}-\bar{h}_{\bar{x}}\bar{w}_{2y})+2\epsilon Ca^{\frac{1}{3}}\bar{h}_{\bar{x}}\bar{h}_{\bar{z}}(\epsilon Ca^{\frac{1}{3}}\bar{w}_{2\bar{z}}-\bar{u}_{2\bar{x}})$$
$$+(\epsilon Ca^{\frac{1}{3}}\bar{u}_{2\bar{z}}+\bar{w}_{2\bar{x}})(\bar{h}_{\bar{x}}^2-\epsilon^2 Ca^{\frac{2}{3}}\bar{h}_{\bar{z}}^2)$$
$$+\epsilon Ca^{-\frac{1}{3}}((\bar{h}_{\bar{z}}\bar{v}_{2\bar{x}}-\bar{h}_{\bar{x}}\bar{v}_{2\bar{z}})]\quad \text{at}\quad \bar{y}=\bar{h}(x,z), \tag{3.7c}$$

$$[Ca^{\frac{2}{3}}(\bar{h}_{\bar{x}}^2+\epsilon^2 Ca^{\frac{2}{3}}\bar{h}_{\bar{z}}^2)\{\bar{h}_{\bar{x}}(\bar{u}_{1\bar{y}}+Ca^{\frac{2}{3}}\bar{v}_{1\bar{x}})+\epsilon Ca^{\frac{1}{3}}\bar{h}_{\bar{z}}(\epsilon Ca\bar{v}_{1\bar{z}}+\bar{w}_{1\bar{y}})\}$$
$$+2Ca^{\frac{2}{3}}\{\bar{h}_{\bar{x}}^2(\bar{u}_{1\bar{x}}-\bar{v}_{1\bar{y}})+\epsilon^2 Ca^{\frac{2}{3}}\bar{h}_{\bar{z}}^2(\epsilon Ca^{\frac{1}{3}}\bar{w}_{1\bar{z}}-\bar{v}_{1\bar{y}})$$
$$+\epsilon Ca^{\frac{1}{3}}\bar{h}_{\bar{x}}\bar{h}_{\bar{z}}(\epsilon Ca^{\frac{1}{3}}\bar{u}_{1\bar{z}}+\bar{w}_{1\bar{x}})\}-\bar{h}_{\bar{x}}(\bar{u}_{1\bar{y}}+Ca^{\frac{2}{3}}\bar{v}_{1\bar{x}})-\epsilon Ca^{\frac{1}{3}}\bar{h}_{\bar{z}}(\epsilon Ca\bar{v}_{1\bar{z}}+\bar{w}_{1\bar{y}})]$$
$$=M[Ca^{\frac{2}{3}}(\bar{h}_{\bar{x}}^2+\epsilon^2 Ca^{\frac{2}{3}}\bar{h}_{\bar{z}}^2)\{\bar{h}_{\bar{x}}(Ca^{\frac{2}{3}}\bar{u}_{2y}+\bar{v}_{2\bar{x}})$$
$$+\epsilon Ca^{\frac{1}{3}}\bar{h}_{\bar{z}}(\epsilon Ca^{\frac{1}{3}}\bar{v}_{2\bar{z}}+Ca^{\frac{2}{3}}\bar{w}_{2y})\}$$
$$+2Ca^{\frac{2}{3}}\{\bar{h}_{\bar{x}}^2(\bar{u}_{2\bar{x}}-\bar{v}_{2y})+\epsilon^2 Ca^{\frac{2}{3}}\bar{h}_{\bar{z}}^2(\epsilon Ca^{\frac{1}{3}}\bar{w}_{2\bar{z}}-\bar{v}_{2y})$$
$$+\epsilon Ca^{\frac{1}{3}}\bar{h}_{\bar{x}}\bar{h}_{\bar{z}}(\epsilon Ca^{\frac{1}{3}}\bar{u}_{2\bar{z}}+\bar{w}_{2\bar{x}})\}$$
$$-\bar{h}_{\bar{x}}(Ca^{\frac{2}{3}}\bar{u}_{2y}+\bar{v}_{2\bar{x}})-\epsilon Ca^{\frac{1}{3}}\bar{h}_{\bar{z}}(\epsilon Ca^{\frac{1}{3}}\bar{v}_{2\bar{z}}+Ca^{\frac{2}{3}}\bar{w}_{2y})]\quad \text{at}\quad \bar{y}=\bar{h}(\bar{x},\bar{z}), \tag{3.7d}$$

$$P_2-P_1=\frac{2Ca^{\frac{2}{3}}}{1+Ca^{\frac{2}{3}}\bar{h}_{\bar{x}}^2+\epsilon^2 Ca^{\frac{4}{3}}\bar{h}_{\bar{z}}^2}[\{\bar{h}_{\bar{x}}(\bar{u}_{1\bar{y}}+Ca^{\frac{2}{3}}\bar{v}_{1\bar{x}})+\epsilon Ca^{\frac{1}{3}}\bar{h}_{\bar{z}}(\epsilon Ca\bar{v}_{1\bar{z}}+\bar{w}_{1\bar{y}})$$
$$-\epsilon Ca\bar{h}_{\bar{x}}\bar{h}_{\bar{z}}(\epsilon Ca^{\frac{1}{3}}\bar{u}_{1\bar{z}}+\bar{w}_{1\bar{x}})-\bar{v}_{1\bar{y}}-Ca^{\frac{2}{3}}\bar{h}_{\bar{x}}^2\bar{u}_{1\bar{x}}-\epsilon^3 Ca^{\frac{5}{3}}\bar{h}_{\bar{z}}^2\bar{w}_{1\bar{z}}\}$$
$$-M\{\bar{h}_{\bar{x}}(Ca^{\frac{2}{3}}\bar{u}_{2y}+\bar{v}_{2\bar{x}})$$
$$+\epsilon Ca^{\frac{1}{3}}\bar{h}_{\bar{z}}(\epsilon Ca^{\frac{1}{3}}\bar{v}_{2\bar{z}}+Ca^{\frac{2}{3}}\bar{w}_{2y})-\epsilon Ca\bar{h}_{\bar{x}}\bar{h}_{\bar{z}}(\epsilon Ca^{\frac{1}{3}}\bar{u}_{2\bar{z}}+\bar{w}_{2\bar{x}})$$
$$-\bar{v}_{2y}-Ca^{\frac{2}{3}}\bar{h}_{\bar{x}}^2\bar{u}_{2\bar{x}}-\epsilon^3 Ca^{\frac{5}{3}}\bar{h}_{\bar{z}}^2\bar{w}_{2\bar{z}}\}]$$
$$+\frac{1}{(1+Ca^{\frac{2}{3}}\bar{h}_{\bar{x}}^2+\epsilon^2 Ca^{\frac{4}{3}}\bar{h}_{\bar{z}}^2)^{\frac{3}{2}}}[\bar{h}_{\bar{x}\bar{x}}(1+\epsilon^2 Ca^{\frac{4}{3}}\bar{h}_{\bar{z}}^2)$$
$$+\epsilon^2 Ca^{\frac{2}{3}}\bar{h}_{\bar{z}\bar{z}}(1+Ca^{\frac{2}{3}}\bar{h}_{\bar{x}}^2)-2\epsilon^2 Ca^{\frac{4}{3}}\bar{h}_{\bar{x}}\bar{h}_{\bar{z}}\bar{h}_{\bar{x}\bar{z}}]\quad \text{at}\quad \bar{y}=\bar{h}(\bar{x},\bar{z}), \tag{3.7e}$$

$$\bar{u}_{2y}=0,\quad \bar{v}_2=0\quad \text{at}\quad y=0. \tag{3.8}$$

y and $\bar{y}$ are unstretched and stretched independent variables in the vertical direction of fluid 2 and 1 respectively. The bar on y for fluid 2 is dropped to distinguish it from that of fluid 1.

4. Low-capillary-number expansion

From the equations and boundary conditions of the two subregions, it can be assumed that all unknown quantities may be expanded in simple powers of ϵ and $Ca^{\frac{1}{3}}$ as follows:

$$\left.\begin{aligned} h(x,z) &= \sum_{i,j=0}^{\infty} \epsilon^i Ca^{\frac{1}{3}j} h^{ij}(x,z), \\ P(x,y,z) &= \sum_{i,j=0}^{\infty} \epsilon^i Ca^{\frac{1}{3}j} p^{ij}(x,y,z), \\ \boldsymbol{u}(x,y,z) &= \sum_{i,j=0}^{\infty} \epsilon^i Ca^{\frac{1}{3}j} \boldsymbol{u}^{ij}(x,y,z). \end{aligned}\right\} \tag{4.1}$$

This choice of gauge functions is justified if matching is possible at each order.

It is useful to develop the expansion procedure in each subregion of region II to lowest order first, as the full development is quite complicated. This has two advantages, the first of which is (we hope) ease of following the development on the part of the reader, and the second is to put the development of Landau & Levich (1942) and of Bretherton (1961) into the modern context of matched asymptotic expansions.

4.1. *The capillary-statics region*

Substituting (4.1) into (3.1)–(3.4), the equations of change and boundary conditions for the capillary-statics region can be derived for each order of ϵ and Ca. For the leading-order approximation in both ϵ and Ca the equation and boundary conditions become

$$u^{00}_{1x}+v^{00}_{1y}=0, \quad P^{00}_{1x}=0, \quad P^{00}_{1y}=0, \tag{4.2a}$$

$$u^{00}_{2x}+v^{00}_{2y}=0, \quad P^{00}_{2x}=0, \quad P^{00}_{2y}=0 \tag{4.2b}$$

(in writing (4.2*b*) we have assumed $M=o(Ca^{-1})$),

$$u^{00}_{1}=-1, \quad v^{00}_{1}=0 \quad \text{at} \quad y=-1, \tag{4.3}$$

$$u^{00}_{1}=u^{00}_{2}, \quad v^{00}_{1}=v^{00}_{2} \quad \text{at} \quad y=-h^{00}(x,z), \tag{4.4a}$$

$$u^{00}_{1}h^{00}_{x}=-v^{00}_{1} \quad \text{at} \quad y=-h^{00}(x,z), \tag{4.4b}$$

$$\{(1-(h^{00}_{x})^{2})(u^{00}_{1y}+v^{00}_{1x})+4h^{00}_{x}u^{00}_{1x}\}$$
$$=M\{(1-(h^{00}_{x})^{2})(u^{00}_{2y}+v^{00}_{2x})+4h^{00}_{x}u^{00}_{2x}\} \quad \text{at} \quad y=-h^{00}(x,z), \tag{4.4c}$$

$$P^{00}_{2}-P^{00}_{1}=\Delta P^{00}=-\frac{h^{00}_{xx}}{(1+(h^{00}_{x})^{2})^{\frac{3}{2}}} \quad \text{at} \quad y=-h^{00}(x,z), \tag{4.4d}$$

$$u^{00}_{2y}=0, \quad v^{00}_{2}=0 \quad \text{at} \quad y=0. \tag{4.5}$$

As we can see, the viscous-force term is negligible in the equation of motion. Therefore we need not yet consider the flow field, and we can obtain the shape of the static meniscus by integrating the normal stress balance, which of course at this order is the Laplace–Young equation. The result is

$$h^{00}(x,z)=\frac{1}{\Delta P^{00}}[1-\{\Delta P^{00}(x-f(z))+1\}^{2}]^{\frac{1}{2}}. \tag{4.6}$$

Two boundary conditions that were used for this integration, and which apply at *all orders* in $Ca^{\frac{1}{3}}$, are

$$\left.\begin{aligned} h_x &\rightarrow -\infty \quad \text{as} \quad x\rightarrow f(z),\\ h &= 0 \quad \text{at} \quad x=f(z). \end{aligned}\right\} \tag{4.7}$$

From (4.6) we see that the leading-order solution for the tip of the boundary is a circle, modulated in the spanwise direction by the function $f(z)$. This solution obviously cannot be uniformly valid, and must be matched to a solution valid in the transition layer. This 'outer' solution contains an unknown constant pressure jump ΔP^{00}, which must be determined by applying matching conditions between the two subregions and will be discussed later. But we can easily anticipate that it will be 1, or, in dimensional terms, the pressure jump across a circular interface with radius b and the interfacial tension γ.

4.2. *The transition region*

In the transition region the equations and boundary conditions for the leading-order approximation can be derived from (3.5)–(3.8) as

$$\bar{u}^{00}_{1\bar{x}}+\bar{v}^{00}_{1\bar{y}} = 0, \quad \bar{P}^{00}_{1\bar{x}} = \bar{u}^{00}_{1\bar{y}\bar{y}}, \quad \bar{P}^{00}_{1\bar{y}} = 0, \tag{4.8a}$$

$$\bar{u}^{00}_{2\bar{x}}+\bar{v}^{00}_{2\bar{y}} = 0, \quad \bar{P}^{00}_{2\bar{x}} = 0, \quad \bar{P}^{00}_{2\bar{y}} = M\bar{v}^{00}_{2\bar{x}\bar{x}} \tag{4.8b}$$

(in writing (4.8*b*) we have assumed $M = o(Ca^{-\frac{2}{3}})$),

$$\bar{u}^{00}_1 = -1, \quad \bar{v}^{00}_1 = 0 \quad \text{at} \quad \bar{y} = 0, \tag{4.9}$$

$$\left.\begin{aligned} \bar{u}^{00}_1 &= \bar{u}^{00}_2, \quad \bar{v}^{00}_2 = 0, & (4.10a)\\ \bar{u}^{00}_1\bar{h}^{00}_{\bar{x}} &= \bar{v}^{00}_1, & (4.10b)\\ \bar{u}^{00}_{1\bar{y}} &= 0, & (4.10c)\\ \bar{P}^{00}_2-\bar{P}^{00}_1 &= \bar{h}^{00}_{\bar{x}\bar{x}} & (4.10d)\end{aligned}\right\} \quad \text{at} \quad \bar{y} = \bar{h}^{00}(\bar{x},\bar{z}),$$

$$\bar{u}^{00}_{2y} = 0, \quad \bar{v}^{00}_2 = 0 \quad \text{at} \quad y = 1. \tag{4.11}$$

By solving these equations and matching to the constant-film thickness solution of Saffman & Taylor (1958) in region I, the velocity field is determined easily and a third-order differential equation for the shape of the interface can be derived from the kinematic condition as

$$\bar{h}^{00}_{\bar{x}\bar{x}\bar{x}} = 3\frac{\bar{h}^{00}-\bar{t}^{00}}{(\bar{h}^{00})^3}. \tag{4.12}$$

For details see Bretherton (1961). $\bar{t}^{00}$ means the leading-order approximation for the constant film thickness, which is also determined by a matching condition. The differential equation (4.12) was encountered by Landau & Levich (1942) and Bretherton (1961), in apparently different problems.

For the numerical integration, (4.12) may be transformed into a canonical form (4.15) by the transformation (4.13) and (4.14):

$$H^{00}(X,\bar{z}) = \frac{\bar{h}^{00}(\bar{x},z)}{\bar{t}^{00}}, \tag{4.13}$$

$$X = \frac{\bar{x}+s}{\bar{t}^{00}}, \tag{4.14}$$

$$H^{00}_{XXX} = \frac{3(H^{00}-1)}{(H^{00})^3}. \tag{4.15}$$

The condition for matching with region I is simply

$$H^{00}\rightarrow 1 \quad \text{as} \quad X\rightarrow -\infty. \tag{4.16}$$

By linearizing (4.15) about $H^{00} = 1$, an analytic solution for H^{00} valid as $X\rightarrow-\infty$ is determined as

$$H^{00}(X,\bar{z}) = 1+A(\bar{z})\exp\{3^{\frac{1}{3}}X\}. \tag{4.17}$$

For more details see Bretherton (1961). Because the arbitrary shift of coordinates s, which may be a function of $\bar{z}$, is introduced, we can pick an arbitrary value for $A(\bar{z})$ without loss of generality. Thus the differential equation can be integrated numerically with boundary conditions set from (4.17) at some finite value of X. Figure 3 shows the function $H^{00}(X)$, determined numerically, together with a function H^{01}

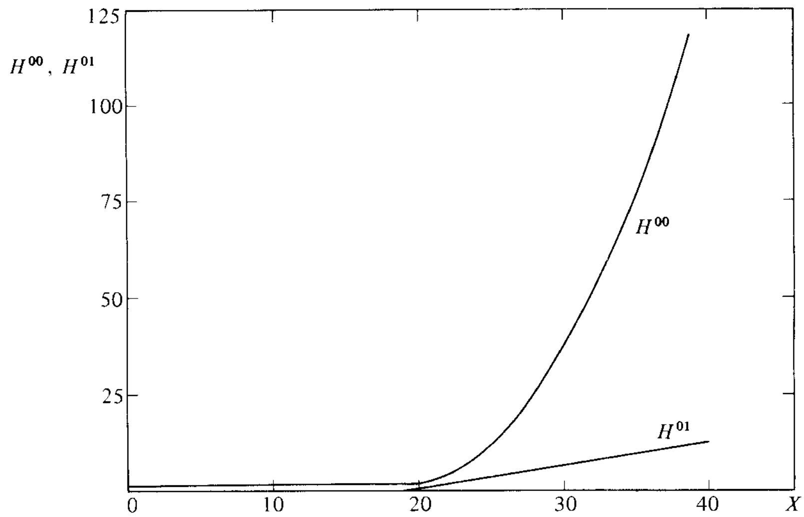

FIGURE 3. The functions $H^{00}(X)$ and $H^{01}(X)$ as determined by numerical integration.

to be discussed below. These numerical results were determined using initial-value techniques and a 4th-order Runge–Kutta routine. The specific values are a result of setting the arbitrary initial conditions (equivalently $A(\bar{z})$ in (4.17)). Because H^{00} goes to infinity as X becomes large, it is possible to show that it has the following quadratic form when X is very large:

$$H^{00}(X,\bar{z}) = \tfrac{1}{2}C_0 X^2 + C_1(\bar{z})\,X + C_2(\bar{z}). \tag{4.18}$$

Therefore, in terms of the original scaling (see (4.13)),

$$\bar{h}^{00}(\bar{x},\bar{z}) = \frac{C_0}{2\bar{t}^{00}}\bar{x}^2 + \left(\frac{C_0 s}{\bar{t}^{00}} + C_1\right)\bar{x} + \left(\frac{C_0 s^2}{2\bar{t}^{00}} + C_1 s + C_2 \bar{t}^{00}\right) \quad \text{as} \quad \bar{x}\to\infty. \tag{4.19}$$

Because C_0 is invariant under a shift along the X-axis, it is not a function of $\bar{z}$. The unknowns $\bar{t}^{00}$ and $s(\bar{z})$ are determined by matching conditions.

4.3. *The matching conditions*

Matching of the solution in the transition layer with that in region I is ensured by (4.16), but is incomplete until the thickness $\bar{t}^{00}$ is determined. This and all other quantities are determined by matching with the solution in the capillary statics region. This matching condition is given by the following simple equation:

$$\lim_{\bar{x}\to\infty}\{1 - Ca^{\frac{2}{3}}\bar{h}(\bar{x},\bar{z})\} = \lim_{x\to -l} h(x,z). \tag{4.20}$$

The limits are interpreted in terms of the matching principle of Van Dyke (1964). By expanding $h(x,z)$ about $x=-l$ using Taylor-series expansion, rewriting the expansion in inner variables, and comparing it with the left-hand side term by term, matching conditions for each order can easily be determined. They are

$O(\epsilon^0 Ca^0)$:

$$h^{00}(-l^0,z) = 1; \tag{4.21}$$

$O(\epsilon^0 Ca^{\frac{1}{3}})$:

$$h_x^{00}(-l^0,z) = 0, \tag{4.22a}$$

$$h^{01}(-l^0,z) = 0; \tag{4.22b}$$

$O(\epsilon^0 Ca^{\frac{2}{3}})$:

$$h^{00}_{xx}(-l^0, z) = -\frac{C_0}{\bar{t}^{00}}, \tag{4.23a}$$

$$h^{01}_{x}(-l^0, z) = -\left(\frac{C_0 s}{\bar{t}^{00}} + C_1\right), \tag{4.23b}$$

$$h^{02}(-l^0, z) = -\left(\frac{C_0 s^2}{2\bar{t}^{00}} + C_1 s + C_2 \bar{t}^{00}\right). \tag{4.23c}$$

Obviously these matching conditions give information on higher-order corrections to the leading-order solution presently under discussion. Extracting those conditions applying only to h^{00}, we have

$$\left.\begin{aligned} h^{00}(-l^0, z) &= 1, \\ h^{00}_{x}(-l^0, z) &= 0, \\ h^{00}_{xx}(-l^0, z) &= -\frac{C_0}{\bar{t}^{00}}. \end{aligned}\right\} \tag{4.24}$$

Physically these conditions state that the outer static solution apparently meets the wall with zero slope (an apparent contact angle of 180°), and that its curvature at that point matches that of the inner transition-layer solution. The first condition serves to locate the origin of the transition layer at the apparent zero of the outer solution. In this sense, the present problem is analogous to those discussed by Ruschak & Scriven (1977) and by Renk, Wayner & Homsy (1978). Using (4.21) and (4.22a), ΔP^{00} and l^0 can be determined:

$$\Delta P^{00} = 1, \tag{4.25a}$$

$$l^0 = 1 - f(z). \tag{4.25b}$$

Because the origin of the coordinate system is moving with the interface, we can anticipate that the origin l, of the transition region is not a function of Ca. This can also be proved by induction. Therefore l has only one superscript, which represents the order of ϵ. Knowing l^0 and ΔP^{00}, the last of (4.24) can be applied, and $\bar{t}^{00}$ is determined as

$$\bar{t}^{00} = C_0. \tag{4.26}$$

$\bar{t}^{00}$ was stretched by $Ca^{\frac{2}{3}}$, and C_0 is known to be 1.337 from the result of numerical integration; therefore the dimensionless film thickness t^{00} is given by

$$t^{00} = 1.337\left(\frac{\mu_1 U}{\gamma}\right)^{\frac{2}{3}}. \tag{4.27}$$

4.4. *Higher-order corrections*

Thus far we have developed a rational expansion procedure, valid for small Ca, and have recovered the previous results of Landau & Levich (1942) and Bretherton (1961). These show that, at leading order in Ca, the tip of the advancing interface is circular in form and, except for a thin film of thickness $O(Ca^{\frac{2}{3}})$, fills the slot with radius 1. Thus the pressure drop across this interface, given by ΔP^{00}, is also 1. In this subsection, we will compute higher-order corrections to this pressure drop and to the film thickness.

Up to $O(Ca^{\frac{2}{3}})$ it is easy to see that the viscous force is negligible, and only the normal stress balance is of importance in the capillary statics region. Thus we have, at $O(\epsilon^0 Ca^{\frac{1}{3}})$,

$$P_2^{01} - P_1^{01} = \Delta P^{01} = -\left[\frac{h_x^{01}}{(1+(h_x^{00})^2)^{\frac{3}{2}}}\right]_x. \qquad (4.28)$$

Integrating (4.28) with the two boundary conditions from (4.7) and applying the matching condition (4.22*b*), ΔP^{01} and $h^{01}(x,z)$ are shown to be zero. This may be seen simply because the $O(Ca^{\frac{1}{3}})$ problem is homogeneous. Therefore from (4.23*b*) the shift $s(\bar{z})$ is determined as

$$s(\bar{z}) = -C_1(\bar{z}). \qquad (4.29)$$

This completes the lowest-order solution as it fixes the location of the solution H^{00} relative to the origin of the transition layer.

Because h^{01} is zero, the normal-stress balance for the next order in Ca can be simplified to

$$P_2^{02} - P_1^{02} = \Delta P^{02} = -\left[\frac{h_x^{02}}{(1+(h_x^{00})^2)^{\frac{3}{2}}}\right]_x \qquad (4.30)$$

at $O(\epsilon^0 Ca^{\frac{2}{3}})$. With reference to equation (3.3*d*), this form is correct if $M = o(Ca^{-\frac{1}{3}})$. This is the most stringent restriction on the viscosity ratio in our theory. Similarly (4.30) can be integrated to give

$$h^{02}(x,z) = \Delta P^{02} \frac{x-f(z)}{\{1-(x-f(z)+1)^2\}^{\frac{1}{2}}}. \qquad (4.31)$$

By applying the matching condition (4.23*c*) to (4.31), ΔP^{02} is determined as

$$\Delta P^{02} = C_2(\bar{z})\, C_0 - \tfrac{1}{2}C_1^2(\bar{z}). \qquad (4.32)$$

Even though C_1 and C_2 are functions of $\bar{z}$, we can see from (4.18) that $C_2(\bar{z})\, C_0 - \frac{1}{2}C_1^2(\bar{z})$ is invariant under the shift of the origin along the X-axis. Therefore ΔP^{02} is not a function of $\bar{z}$ but a constant, and the numerical value is approximately 3.80 (Ruschak 1974). Therefore the pressure jump ΔP is given by

$$\Delta P = 1 + 3.80 Ca^{\frac{2}{3}} + O(Ca, \epsilon). \qquad (4.33)$$

This result verifies Bretherton's result (Bretherton 1961, p. 172) in a rigorous manner by applying the full matching conditions.

In the transition region, the equation of motion for $O(\epsilon^0 Ca^{\frac{1}{3}})$ is still a lubrication approximation and independent of M if we apply the earlier restriction that $M = o(Ca^{-\frac{1}{3}})$. Therefore it can be easily solved to give the following third-order differential equation for $\bar{h}^{01}$:

$$\bar{h}_{\overline{xxx}}^{01} = -3 \frac{\bar{h}^{01}(2\bar{h}^{00} - 3\bar{t}^{00}) + \bar{t}^{01}\bar{h}^{00}}{(\bar{h}^{00})^4}. \qquad (4.34)$$

$\bar{t}^{01}$ represents the $O(\epsilon^0 Ca^{\frac{1}{3}})$ correction term to the film thickness of region I. Equation (4.34) can be transformed into a canonical form by a transformation similar to (4.13) and (4.14). Thus we find

$$H_{XXX}^{01}(X,z) = -3\frac{H^{01}(2H^{00}-3) + R(\bar{z})\, H^{00}}{(H^{00})^4}, \qquad (4.35)$$

with matching condition

$$H^{01} \to R(\bar{z}), \quad H^{00} \to 1 \quad \text{as} \quad X \to -\infty, \qquad (4.36)$$

where $R(\bar{z}) = \bar{t}^{01}/\bar{t}^{00}$.

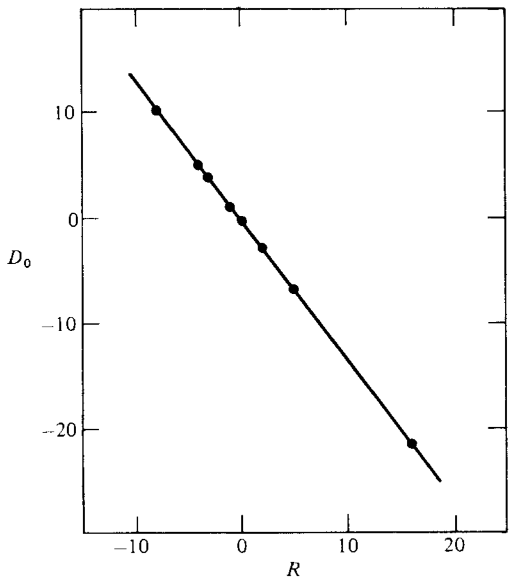

FIGURE 4. The dependence of D_0 upon R as determined by numerical integration of (4.35) for various values of R. The solid dots are computed points.

Using the same procedure as at lower orders, the analytic solution for H^{01} valid as $X \to -\infty$ is determined as

$$H^{01}(X, \bar{z}) = R(\bar{z}) + B(\bar{z})\, e^{3^{\frac{1}{3}}X}. \tag{4.37}$$

Since $H^{00} \to \infty$ as $x \to \infty$, (4.35) shows that H^{01}_{XXX} approaches zero as X becomes large. Therefore the behaviour of H^{01} as $X \to \infty$ is also expressed as the following quadratic equation:

$$H^{01}(X, \bar{z}) = \tfrac{1}{2}D_0(\bar{z})\, X^2 + D_1(\bar{z})\, X + D_2(\bar{z}). \tag{4.38}$$

Therefore

$$\bar{h}^{01}(\bar{x}, \bar{z}) = \frac{D_0}{2\bar{t}^{00}}\bar{x}^2 + \left(\frac{D_0 s}{\bar{t}^{00}} + D_1\right)\bar{x} + \frac{D_0 s^2}{2\bar{t}^{00}} + D_1 s + D_2 \bar{t}^{00}. \tag{4.39}$$

Because D_0 is invariant under the shift along the X-axis, $D_0(\bar{z})$ depends on $R(\bar{z})$ only. But $D_1(\bar{z})$ and $D_2(\bar{z})$ depend on both $R(\bar{z})$ and $B(\bar{z})$. The matching conditions at $O(\epsilon^0)$ and $O(Ca)$ may be obtained by applying the matching principle to (4.20) at higher order:

$$h^{01}_{xx}(-l^0, z) = -\frac{D_0}{\bar{t}^{00}}, \tag{4.40a}$$

$$h^{02}_{x}(-l^0, z) = -\left(\frac{D_0 s}{\bar{t}^{00}} + D_1\right), \tag{4.40b}$$

$$h^{03}(-l^0, z) = -\left(\frac{D_0 s^2}{2\bar{t}^{00}} + D_1 s + D_2 \bar{t}^{00}\right) \tag{4.40c}$$

at $O(\epsilon^0 Ca)$. Because h^{01} is zero everywhere, D_0 must be zero. Therefore we must find that value of R which makes D_0 zero, and thus the matching condition may be thought of as providing a boundary condition for (4.35).

Figure 4 shows the dependence of D_0 upon R, as determined by numerical integration of (4.35) for various values of R.

Since the outer solution $h^{02}(x,z)$, s, $\bar{t}^{00}$ and D_0 are all known, $D_1(\bar{z})$ can be determined from (4.40*b*). Therefore $B(\bar{z})$ and $D_2(\bar{z})$ can also be determined through the numerical integration for H^{01}. Because $R(\bar{z})$ is zero we have

$$\bar{t}^{01}(\bar{z}) = 0. \tag{4.41}$$

This tells us that the film thickness of region I does not have an $O(\epsilon^0 Ca^{\frac{1}{3}})$ correction, even though the profile of the transition region does have a correction at this order. This correction, denoted as H^{01}, is given in figure 3. It represents a slight adjustment to the thickness profile in the transition region, but, because the first correction to ΔP in the capillary-statics region is $O(Ca^{\frac{2}{3}})$, this profile cannot have any curvature as $\bar{x}\rightarrow\infty$. The first correction to the film thickness will also be $O(Ca^{\frac{2}{3}})$, i.e. $\bar{t}^{02}$ will be non-zero, but we shall not pursue its calculation here.

5. Small-ϵ expansion

So far we have only discussed higher-order correction terms in an expansion in $Ca^{\frac{1}{3}}$, which are induced by the action of viscous forces within the transition layer.

We now wish to investigate the higher-order correction terms in ϵ, which will give us the correction that is set up by the z-variation of the interface. Because the viscous-force term is negligible in the equations of motion for the capillary-statics region through $O(Ca^{\frac{2}{3}})$ regardless of ϵ, the normal stress balance is still the only important condition. The normal stress balance at $O(\epsilon Ca^0)$ can be simplified as

$$P_2^{10}-P_1^{10} = \Delta P^{10} = -\left[\frac{h_x^{10}}{(1+(h_x^{00})^2)^{\frac{3}{2}}}\right]_x. \tag{5.1}$$

It can be integrated with the two boundary conditions from (4.7) to give

$$h^{10}(x,z) = \Delta P^{10}\frac{x-f(z)}{\{1-(x-f(z)+1)^2\}^{\frac{1}{2}}}. \tag{5.2}$$

The matching conditions for $O(\epsilon)$ are given by

$O(\epsilon Ca^0)$:

$$h^{10}(-l^0,z) = 0; \tag{5.3}$$

$O(\epsilon Ca^{\frac{1}{3}})$:

$$h^{11}(-l^0,z) = 0, \tag{5.4a}$$

$$l^1(z)\,h_{xx}^{00}(-l^0,z) = 0; \tag{5.4b}$$

$O(\epsilon Ca^{\frac{2}{3}})$:

$$h^{12}(-l^0,z)+h_x^{11}(-l^0,z)\,\bar{x}+\tfrac{1}{2}h_{xx}^{10}(-l^0,z)\,\bar{x}^2 = -\bar{h}^{10}(\infty,\bar{z}). \tag{5.5}$$

ΔP^{10} is shown to be zero from (5.3), thus $h^{10}(x,z)$ is zero everywhere.

With this result, the normal stress balance at the next order in $Ca^{\frac{1}{3}}$ is given by

$$P_2^{11}-P_1^{11} = \Delta P^{11} = -\left[\frac{h_x^{11}}{(1+(h_x^{00})^2)^{\frac{3}{2}}}\right]_x \tag{5.6}$$

at $O(\epsilon Ca^{\frac{1}{3}})$. From the matching condition (5.4*a*) it is obvious that ΔP^{11} and $h^{11}(x,z)$ are also zero.

In the transition region at $O(\epsilon Ca^0)$ the equations of change can be solved to give another third-order differential equation for the shape of the interface as follows:

$$\bar{h}_{\bar{x}\bar{x}\bar{x}}^{10}(\bar{x},\bar{z}) = -3\frac{\bar{h}^{10}(2\bar{h}^{00}-3\bar{t}^{00})+\bar{t}^{10}\cdot\bar{h}^{00}}{(\bar{h}^{00})^4}. \tag{5.7}$$

This differential equation is exactly the same as (4.34). Therefore, by following the same procedure along with the matching condition (5.5) applied to the asymptotic form of $\bar{h}^{10}$, $\bar{t}^{10}$ can be determined to be zero.

Proceeding to $O(\epsilon^2)$, the extremely complicated normal stress balance (3.3*d*) for the capillary-statics region at $O(\epsilon^2 Ca^0)$ can be simplified considerably, since $h^{01}(x,z)$ is known to be zero. Thus we find, at $O(\epsilon^2 Ca^0)$,

$$P_2^{20} - P_1^{20} = \Delta P^{20} = -\frac{1}{(1+(h_x^{00})^2)^{\frac{5}{2}}}[\{h_{xx}^{20}(1+(h_x^{00})^2) - 3h_{xx}^{00}\,h_x^{00}\,h_x^{20}\} + \{(h_z^{00})^2\,h_{xx}^{00}(1+(h_x^{00})^2) + h_{zz}^{00}(1+(h_x^{00})^2)^2 - 2h_x^{00}\,h_z^{00}\,h_{xz}^{00}(1+(h_x^{00})^2) - \tfrac{3}{2}h_{xx}^{00}(h_z^{00})^2\}], \tag{5.8}$$

which may be written

$$\Delta P^{20} = -\left[\frac{h_x^{20}}{(1+(h_x^{00})^2)^{\frac{3}{2}}}\right]_x - \frac{1}{(1+(h_x^{00})^2)^{\frac{5}{2}}}\{(h_x^{00})^2\,h_{xx}^{00}(1+(h_x^{00})^2) + h_{zz}^{00}(1+(h_x^{00})^2)^2 - 2h_z^{00}h_x^{00}h_{xz}^{00}(1+(h_x^{00})^2) - \tfrac{3}{2}h_{xx}^{00}(h_z^{00})^2\}. \tag{5.9}$$

Because $h^{00}(x,z)$ is known as the family of modulated circles ((4.6) with $\Delta P^{00} = 1$), (5.9) may be further simplified to read

$$\Delta P^{20} = -\left[\frac{h_x^{20}}{(1+(h_x^{00})^2)^{\frac{3}{2}}}\right]_x - f''(x-f+1) + (f')^2\{1 - \tfrac{3}{2}(x-f+1)^2\}. \tag{5.10}$$

Here the prime refers to differentiation with respect to z. With the two boundary conditions from (4.7), (5.10) can be integrated to give

$$h^{20}(x,z) = \{\Delta P^{20} - \tfrac{1}{2}(f')^2\}\frac{x-f}{\{1-(x-f+1)^2\}^{\frac{1}{2}}} + \tfrac{1}{2}f''\sin^{-1}(x-f+1) - \tfrac{1}{2}(f')^2\{1-(x-f+1)^2\}^{\frac{1}{2}} - \tfrac{1}{4}\pi f''. \tag{5.11}$$

The matching conditions at $O(\epsilon^2)$ are given by

$O(\epsilon^2 Ca^0)$:

$$h^{20}(-l^0, z) = 0; \tag{5.12}$$

$O(\epsilon^2 Ca^{\frac{1}{3}})$:

$$h^{21}(-l^0, z) = 0, \tag{5.13a}$$

$$h_x^{20}(-l^0, z) - l^{(2)}h_{xx}^{00}(-l^0, z) = 0; \tag{5.13b}$$

$O(\epsilon^2 Ca^{\frac{2}{3}})$:

$$\{h^{22}(-l^0,z) - l^{(2)}h_x^{02}(-l^0,z)\} + h_x^{21}(-l^0,z)\,\bar{x} + \tfrac{1}{2}h_{xx}^{20}(-l^0,z)\,\bar{x}^2 = -\bar{h}^{20}(\infty,\bar{z}). \tag{5.14}$$

ΔP^{20} can thus be determined by applying (5.12):

$$\Delta P^{20} = -\tfrac{1}{4}\pi f''. \tag{5.15}$$

This completes the solution in the capillary-statics region through $O(\epsilon^2)$.

We have determined the order of the correction to (5.15) as follows: The normal stress balance in the capillary statics region has been expanded to $O(\epsilon^2 Ca^{\frac{1}{3}})$. Because h^{01}, h^{10} and h^{11} are known to be zero, the resulting complicated equation can be simplified to give

$$P_2^{21} - P_1^{21} = \Delta P^{21} = -\left[\frac{h_x^{21}}{(1+(h_x^{00})^2)^{\frac{3}{2}}}\right]_x \tag{5.16}$$

at $O(\epsilon^2 Ca^{\frac{1}{3}})$. It is obvious from the matching condition (5.13*a*) that ΔP^{21} and $h^{21}(x,z)$ are zero. At $O(\epsilon^2 Ca^{\frac{2}{3}})$ it is also obvious that *neither* h^{22} nor ΔP^{22} is zero, since $h^{22}(-l^0, z)$ is not zero from (5.14). Thus the correction is $O(\epsilon^2 Ca^{\frac{2}{3}})$.

From these results, ΔP is given by

$$\Delta P = 1+3.80 Ca^{\frac{2}{3}} - \tfrac{1}{4}\pi f'' \epsilon^2 + O(Ca, \epsilon^2 Ca^{\frac{2}{3}}). \tag{5.17}$$

As for the transition region at $O(\epsilon^2 Ca^0)$, the solution for the velocity field and kinematic condition generate another third-order differential equation (5.18), which in principle gives us the correction term to the film thickness set up by the z-variation of the interface:

$$\bar{h}^{20}_{\bar{x}\bar{x}\bar{x}}(\bar{x},\bar{z}) = -3\frac{\bar{h}^{20}(2\bar{h}^{00}-3\bar{t}^{00})+\bar{t}^{20}\bar{h}^{00}}{(\bar{h}^{00})^4}. \tag{5.18}$$

This differential equation is also the same as (4.34). Therefore, by following the same procedure, $\bar{h}^{20}$ can be given for large $\bar{x}$ as

$$\bar{h}^{20}(\bar{x},\bar{z}) = \frac{\bar{D}_0(\bar{z})}{2\bar{t}^{00}}\bar{x}^2 + \left(\frac{\bar{D}_0 s}{\bar{t}^{00}} + \bar{D}_1(\bar{z})\right)\bar{x} + \frac{\bar{D}_0 s^2}{2\bar{t}^{00}} + \bar{D}_1 s + \bar{D}_2(\bar{z})\bar{t}^{00} \quad \text{as} \quad \bar{x}\to\infty. \tag{5.19}$$

From the matching condition (5.14),

$$h^{20}_{xx}(-l^0, z) = -\frac{\bar{D}_0}{\bar{t}^{00}}. \tag{5.20}$$

Therefore

$$\bar{D}_0(\bar{z}) = -\bar{t}^{00}\{\tfrac{1}{4}\pi f'' + (f')^2\}. \tag{5.21}$$

If the relation between $\bar{D}_0$ and R $(=\bar{t}^{20}/\bar{t}^{00})$ is known, $\bar{t}^{20}$ can be determined without specifying $f(z)$. As a result of numerical integration figure 4 shows that $\bar{D}_0$ is related to R by the following equation:

$$\bar{D}_0(\bar{z}) = -1.337 R(\bar{z}). \tag{5.22}$$

Therefore $\bar{t}^{20}$ can be determined explicitly as

$$\bar{t}^{20} = \frac{1}{1.337}(\bar{t}^{00})^2\{\tfrac{1}{4}\pi f'' + (f')^2\}. \tag{5.23}$$

$\bar{t}^{00}$ is known from §4; therefore

$$\bar{t}^{20} = 1.337\{\tfrac{1}{4}\pi f'' + (f')^2\}. \tag{5.24}$$

Therefore from these results the dimensionless film thickness t is given by

$$t = 1.337[1+\{\tfrac{1}{4}\pi f'' + (f')^2\}\epsilon^2]Ca^{\frac{2}{3}} + O(Ca^{\frac{4}{3}}, \epsilon^2 Ca). \tag{5.25}$$

This result tells us that the film thickness of region I has a z-variation at $O(\epsilon^2 Ca^{\frac{2}{3}})$. It can be proved that the solution of region I itself admits the z-variation of the film thickness to that order, so that no inconsistency is present.

6. Summary and conclusions

In this paper we have presented a theory of two-phase displacement in a Hele Shaw cell which is valid asymptotically in the limit of slow flow. In developing this theory, we have made the assumptions that both the capillary number and the lateral variations in the interfacial position (relative to the gap spacing) are small quantities. It is furthermore necessary to assume that the displaced fluid wets the walls of the gap; otherwise a moving contact line is present, and a suitable model must be proposed for describing its behaviour. Given these assumptions, we have developed the solution as an asymptotic double expansion in the small parameters ϵ and Ca. The expansion in ϵ is found to be uniform while that in Ca is not; the small-Ca limit has been formulated as a matched asymptotic expansion. The results are seen to be independent of the viscosity ratio M as long as $M = o(Ca^{-\frac{1}{3}})$. This formulation allows a rational expansion to be carried out, and earlier work by Landau & Levich (1942) and Bretherton (1961), which is obscure at points, is clarified. In addition, new results involving the lateral variation and higher-order corrections to Bretherton's results are obtained.

Our formulation of the matching procedure shows clearly how the origin of the transition-layer profile may be unambiguously determined, and further shows how the *ad hoc* matching applied by Landau & Levich and Bretherton is correct. In particular, it shows how the curvature in the capillary statics region is determined as a result of matching with the transition layer, whose location is in turn determined as being near the apparent zero of the outer solution. It also shows how the dynamic contact angle under wetting conditions may be developed as an expansion in capillary number.

Our final results may be summarized as follows. For the pressure jump across the interface, we find

$$\frac{\Delta P}{\gamma/b} = 1+3.80Ca^{\frac{2}{3}}-\tfrac{1}{4}\pi f''\epsilon^2+O(Ca, \epsilon^2 Ca^{\frac{2}{3}}). \tag{6.1}$$

We have justified the $O(Ca^{\frac{2}{3}})$ term in this expression by showing that, although there is a change in profile at $O(Ca^{\frac{1}{3}})$, it contributes no change in the tip curvature, and thus the $O(Ca^{\frac{2}{3}})$ term may be computed as the pressure loss for flow in the transition layer, computed from the *lowest-order solution*. The $O(\epsilon^2)$ term of (6.1) is new, and provides the leading-order boundary condition for problems with transverse curvature. As explained in §1, the form of this boundary condition had only been speculated upon previously; our work gives for the first time the explicit relationship. Finally we have shown that the film thickness is given by

$$t = 1.337Ca^{\frac{2}{3}}\{1+\epsilon^2(\tfrac{1}{4}\pi f''+(f')^2)\}+O(\epsilon^2 Ca, Ca^{\frac{4}{3}}), \tag{6.2}$$

and that, unlike the coating-flow problem discussed by Wilson (1982), the $O(Ca^{\frac{1}{3}})$ correction to the film thickness is zero.

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